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Dynamic Mode Decomposition and Proper Orthogonal Decomposition of flow in a lid-driven cylindrical cavity

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ABSTRACT

The flow in a cylindrical cavity driven by a rotating lid undergoes a sequence of bifurcations as the Reynolds number is increased. It is a well-known and well-studied fluid system and a common benchmark for the identification of emerging coherent structures and for the quantification of bifurcation points. Time-resolved particle-image velocimetry (PIV) data have been taken in a cross-sectional plane, and a sequence of snapshots has then been processed by two algorithms: the Dynamic Mode Decomposition (DMD) and the Proper Orthogonal Decomposition (POD).

1. INTRODUCTION

The decomposition of complex flow problems into coherent structures is a common technique to reduce a complicated flow behavior to a simpler one and to gain insight into its underlying mechanisms. The manner in which this decomposition is accomplished and the exact definition of coherence is equally dependent on the flow configuration as well as the specifics of the relevant structures to be extracted (e.g. vortical structures, acoustic waves, mixing indicators).

In this article we will concentrate on two complementary manners of decomposing the flow fields from experimental measurements into coherent structures: the proper orthogonal decomposition (POD) and the Dynamic Mode Decomposition (DMD).

The proper orthogonal decomposition (POD), also referred to in other fields of application as the principal component analysis (PCA), the empirical orthogonal function (EOF) analysis, or the Karhunen-Loève (KL) decomposition, has its origins in signal processing, made its way into fluid dynamics applications in the seventies and eighties [7, 14, 2] and is now a common and standard tool to analyze fluid fields (measured or computed). Over the years, various improvements have been made such as the snapshot technique [14] or the generalization to the bi-orthogonal proper decomposition (BOD) [1, 5]. In its essential form the proper orthogonal decomposition uses the spatial (or temporal) correlation matrix and computes its eigenfunctions, thus decorrelating the structures contained in the snapshots. The extracted structures (eigenfunctions) are interpreted as the building blocks of the analyzed flow but, in addition, are often used as a Galerkin basis for model reduction or control efforts, even in the nonlinear regime [9].

The Dynamic Mode Decomposition (DMD) is a recent extension of the classical Arnoldi technique [3, 4, 15, 6] to accommodate a data-based rather than a model-based framework [13, 12, 11]. A high-degree polynomial is fit to a Krylov sequence of flow fields, which are assumed to become linearly dependent after a sufficient number of snapshots have been taken. When this point is reached, a general linear dependence among the snapshots is assumed and the (unknown) snapshot-to-snapshot mapping is expressed within the snapshot basis. The optimal linear combination of the snapshots is equivalent to a low-dimensional representation of the system dynamics, and all further analysis (such as stability and receptivity computations) can be applied to it.

The difference between the proper orthogonal decomposition and the Dynamic Mode Decomposition lies, respectively, in the use and lack of an averaging step to process the data. In the former case, the POD is based on a time-averaged spatial correlation matrix, whereas in the latter case the DMD approximates the temporal dynamics by a high-degree polynomial.

After more background on either method, we will employ the POD- and DMD-analysis to time-resolved particle-image velocimetry data from flow in a lid-driven cylindrical cavity.

2. BACKGROUND

Starting point for either the POD- or DMD-analysis is a temporal sequence of N data fields $\mathbf{v}_j$ written as

$$\mathbf{V}_1^N = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_N\}. \tag{1}$$

The spacing between each snapshot in the above sequence is assumed to be constant. Each flow field $\mathbf{v}_j$ could contain scalar or vector data in any number of spatial dimensions. Furthermore, subdomains of the entire fluid domain may be treated in isolation, i.e., without regard for the dynamics in the rest of the experimental or computational domain.

The proper orthogonal decomposition proceeds by forming the spatial correlation matrix $\mathbf{C}$ from the above sequence $\mathbf{V}_1^N$ according to

$$\mathbf{C} = \frac{1}{N}\mathbf{V}_1^N \left(\mathbf{V}_1^N\right)^H. \tag{2}$$

More complex integration schemes can be used but do not add significantly to what follows below. The correlation matrix is then decomposed into its eigenvalues λ and eigenvectors Φ

$$\mathbf{C}\Phi_j = \lambda_j \Phi_j. \tag{3}$$

Due to the symmetry of the correlation matrix $\mathbf{C}$ the eigenvalues

λ_j are real and describe the energy content of the coherent flow structure represented by the eigenvector Φ_j (see [7]). For the same reason, the set of eigenvectors is orthogonal which means that each POD-mode Φ_j is statistically decorrelated from any other.

If temporal (rather than spatial) coherence is required, the temporal correlation matrix $\tilde{\mathbf{C}} = \frac{1}{M}(\mathbf{V_1^N})^H \mathbf{V}_1^N$ (with M as the number of degrees of freedom in space) is formed and decomposed accordingly [1, 5]. Both points of view can be addressed by a singular value decomposition of $\mathbf{V}_1^N$ where the spatially coherent structures are given by the right singular vectors and the temporally coherent structures by the left singular vectors.

As mentioned above, the Dynamic Mode Decomposition (DMD) uses the classical Arnoldi idea to express the mapping underlying the snapshot basis

$$\mathbf{v}_{j+1} = \mathbf{A}\mathbf{v}_j \tag{4}$$

by a linear combination of the available data fields $\mathbf{V}_1^{N-1}$. Mathematically, this translates into the approximate matrix equation

$$\mathbf{A}\mathbf{V}_1^{N-1} \approx \mathbf{V}_1^{N-1}\mathbf{S} \tag{5}$$

which is reminiscent of the Arnoldi decomposition where the action of $\mathbf{A}$ on an orthonormalized basis is given by the product of the same basis and a Hessenberg matrix. In our case, we solve directly the equation above, taking advantage of the fact that $\mathbf{S}$ is of companion type [10]. The eigenvalues of $\mathbf{S}$ (also known as Ritz values) approximate some of the eigenvalues of $\mathbf{A}$; the eigenvectors of $\mathbf{S}$ contain the coefficients for the reconstruction of dynamic modes expressed within the snapshot basis.

It is important to realize that the Dynamic Mode Decomposition does not contain any averaging process, neither in time nor in space. As a consequence, the temporal and spatial information is fully preserved. In the formulation above, the temporal information is given by the real and imaginary part of the eigenvalues of $\mathbf{S}$, with the real part as the growth/decay rate and the imaginary part as the frequency of the associated dynamic mode. The spatial information is contained in the corresponding dynamic mode, i.e., the product of the snapshot basis and the chosen eigenvector of $\mathbf{S}$.

Since the snapshot-to-snapshot mapping $\mathbf{A}$ is never formed but rather identified from the data sequence, a simple rearrangement of the flow fields can yield a spatial rather than a temporal analysis. In this case, dynamic modes that evolve, e.g., in the streamwise direction according to a linear mechanism are detected by the Dynamic Mode Decomposition. For further details, the reader is referred to [11]. Moreover, flow structures can also be extracted from image-based flow visualizations (such as a sequence of Schlieren images); see [12].

3. FLOW IN A LID-DRIVEN CYLINDRICAL CAVITY

Flow of an incompressible fluid in a lid-driven cylindrical cavity is considered [8]. The velocity field in a cross-sectional plane is measured using particle-image velocimetry (see Figure 1 for a sketch of the flow configuration and the experimental setup).

The two techniques introduced above will then be applied to a temporally equispaced sequence of 100 particle-image velocimetry (PIV) samples taken during 41 experiments ranging from $Re = 2714$ to $Re = 6333$. Within this range, numerous bifurcations are observed. In particular, the vortex core undergoes a sequence of instabilities, and axial vortices confined to the outer wall regions are common features of this type of flow.

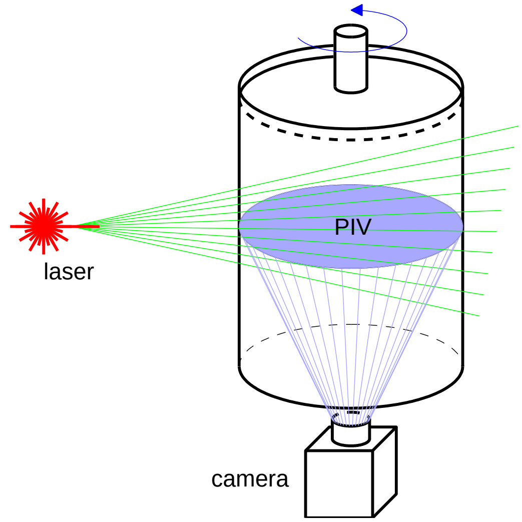

Figure 1: Sketch of geometry and experimental setup of flow in a lid-driven cylindrical cavity.

The results of the two decompositions are shown in Figure 2 where the left column contains the most dominant POD-mode (after the mean flow) and the right column depicts the least stable dynamic mode (again, besides the mean flow). The chosen Reynolds numbers (from top to bottom) are $Re = 4433, 5067, 5429, 5971, 6333$.

Marked differences are visible for the two types of flow decompositions. The POD technique mainly favors the vortical flow in the center of the cavity where the axial mean vortex is prone to vortex-breakdown instabilities. The DMD analysis, on the other hand, extracts in most cases the axial vortices near the cylinder walls. It is clear that the two decomposition techniques emphasize different flow quantities in their identification of coherent structures. Whereas the proper orthogonal decomposition puts a heavy weight on the energy content of each structure and thus ranks the flow in the center of the cavity as more coherent than the axial vortices near the wall (which appear as higher-order POD-modes), the Dynamic Mode Decomposition (DMD) identifies the structures that are most persistent in their dynamics over the observed time horizon. This latter ranking is independent of the energy content of the identified structures.

To recover the temporal dynamics of the various POD-modes a projection of the full dynamics contained in the snapshot sequence onto one or a collection of POD-modes is commonly applied. The coefficients can then be analzyed using Fourier analysis or other, more sophisticated spectral techniques.

Scanning through the 41 data sets ranging from $Re = 2715$ to $Re = 6333$ we can perform a more comprehensive parameter study and plot the frequencies of the most dominant and coherent dynamic modes resulting from a DMD-analysis of respective sequences of 100 snapshots. The results are shown in Figure 3. A marked alignment of the frequencies of the identified dynamic modes along lines can be observed.

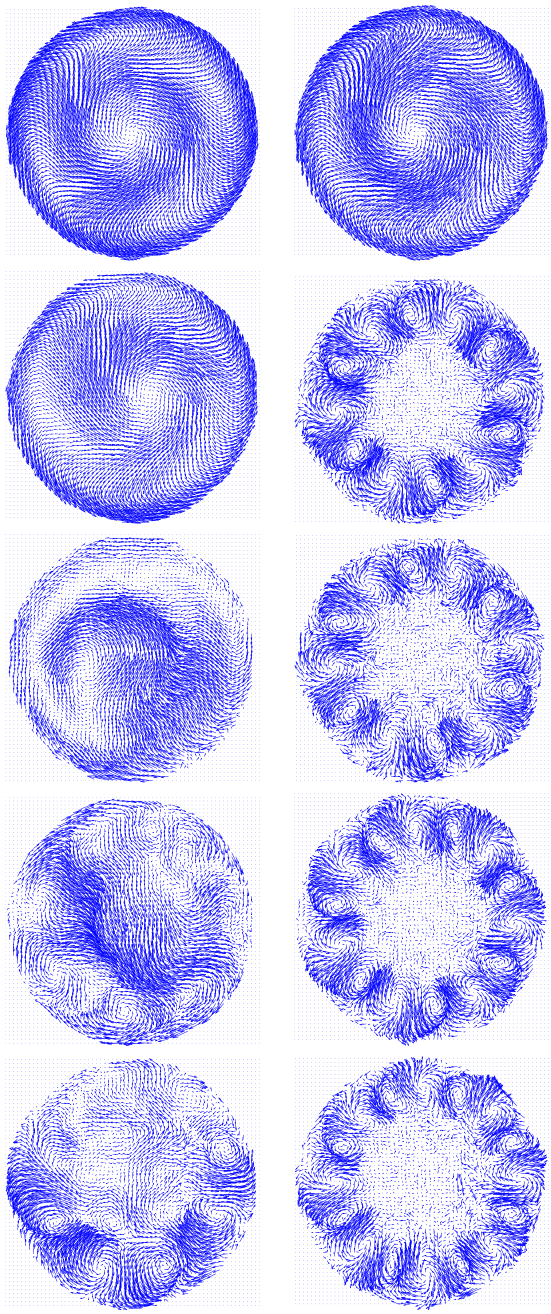

Figure 2: Left column: dominant modes from a proper orthogonal decomposition (POD); right column: least stable mode from a Dynamic Mode Decomposition. The Reynolds numbers, from top to bottom, are $Re = 4433,\ 5067,\ 5429,\ 5971,\ 6333$.

The least stable of the multiple branches is indicated in red. A clear cascade into higher frequencies is noticeable, which appear at discrete Reynolds numbers. The two lowest branches are associated with coherent structures in the center of the cavity related to the dynamics of the vortex core. Higher branches (at higher frequencies) correspond to axial vortices confined to the cylinder wall (as shown in the right column of Figure 2). The appearance of new branches as the Reynolds number is increased indicates bifurcation points in the frequency-Reynolds number parameter space. An analysis of a correspondance of the above results with alternative techniques

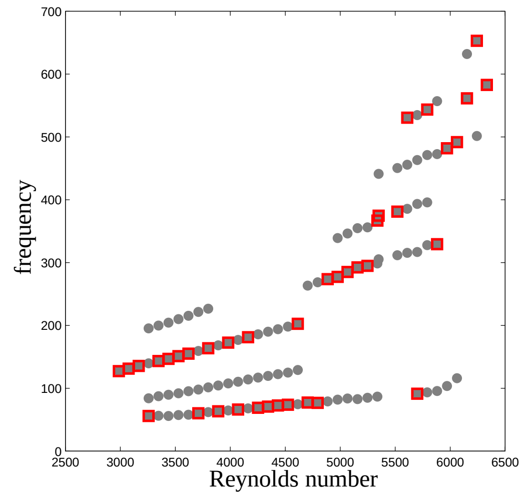

Figure 3: Frequency of the least stable coherent dynamic modes as a function of Reynolds number for flow in a lid-driven cylindrical cavity. The least stable dynamic mode is indicated in red.

and numerical computations is left for a future effort.

4. CONCLUSIONS

The flow in a lid-driven cylindrical cavity exhibits a complex bifurcation behavior characterized by the appearance of vortical motion in its center region and near to its wall. An experiment has been set up to establish a data base of time-resolved particle-image velocimetry (PIV) sequences for a wide range of Reynolds numbers covering multiple bifurcations.

Two complementary decomposition techniques have been applied to the data: the proper orthogonal decomposition (POD) based on the spatial correlation matrix gathered from the snapshots, and the Dynamic Mode Decomposition (DMD) extracting a low-dimensional evolution matrix from the temporally equi-spaced data sequence.

Both techniques identify bifurcation points (i.e., the appearance of new coherent patterns) as the Reynolds number is increased. Whereas the POD concentrates on the more energetic structures of the flow (mainly in the center of the cavity), the DMD-technique isolates the less-energetic but more unstable axial vorticity patterns close to the outer wall of the cylinder. A bifurcation chart, plotting the first few least stable dynamic modes, confirms the discontinuous appearance of higher frequencies associated with modified spatial structures.

This study demonstrates the complementarity and the differences of the two decomposition techniques and introduces the Dynamic Mode Decomposition (DMD) as a new quantitative flow analysis tool to assess the dynamic behavior of fluid flows represented by experimentally measured (or numerically determined) data sequences.

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